

PROPER ISOMETRIC ACTIONS OF HYPERBOLIC GROUPS ON L^p -SPACES

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ABSTRACT. According to a result of G. Yu, any hyperbolic group Γ admits a proper isometric action on $\ell^p(\Gamma \times \Gamma)$ for large enough p . We show that any non-elementary hyperbolic group Γ admits a proper affine isometric action on $L^p(\partial\Gamma \times \partial\Gamma)$, where $\partial\Gamma$ denotes the boundary of Γ , for large enough p . As a key ingredient in our construction, we introduce a Γ -invariant measure on $\partial\Gamma \times \partial\Gamma$ analogous to the Bowen - Margulis measure from the CAT(-1) setting. We also deduce that Γ admits a proper affine isometric action on the first ℓ^p -cohomology group $H_{(p)}^1(\Gamma)$ for large enough p .

1. INTRODUCTION

With respect to the geometry of L^2 -spaces, the class of hyperbolic groups appears to be both “soft” and “rigid”. This ambivalence is vividly illustrated by cocompact lattices in isometry groups of rank-1 symmetric spaces. For real or complex hyperbolic spaces, cocompact lattices admit proper isometric actions on Hilbert spaces. For quaternionic hyperbolic spaces or the octonionic hyperbolic plane, every isometric action of a cocompact lattice on a Hilbert space is bounded. In other words, cocompact lattices enjoy the Haagerup property, or a-T-menability, in the real or complex case (see [7]), respectively Kazhdan’s property (T), or Serre’s property FH, in the quaternionic or octonionic case (see [2]).

When tested against general L^p -spaces, hyperbolic groups reveal themselves to be “soft”: every hyperbolic group admits a proper isometric action on an L^p -space, where p depends on the group. This fact, due to G. Yu, is one of the most interesting results in the study of isometric group actions on uniformly convex Banach spaces. More precisely, the following is shown in [19]:

Theorem 1.1 (Yu). *Let Γ be a hyperbolic group. Then, for large enough p , the linear isometric action of Γ on $\ell^p(\Gamma \times \Gamma)$ admits a proper cocycle.*

Our aim is to prove an analogue of Yu’s result, in which the representation of a hyperbolic group comes from the action on its boundary rather than on itself. Namely:

Theorem 1.2. *Let Γ be a non-elementary hyperbolic group and let $\partial\Gamma$ denote its boundary. Then there is an explicit Γ -invariant measure on $\partial\Gamma \times \partial\Gamma$, with respect to which the following holds: for large enough p , the linear isometric action of Γ on $L^p(\partial\Gamma \times \partial\Gamma)$ admits a proper cocycle.*

The elementary hyperbolic groups are the virtually cyclic ones. Their boundary is essentially trivial, and certainly useless for our purposes. Proper isometric actions of elementary hyperbolic groups on L^p -spaces are easy to construct by other, more direct means.

Although Theorem 1.2 conveys the same idea as Theorem 1.1, namely that hyperbolic groups admit proper isometric actions on L^p -spaces, the two theorems are altogether different both in statement and in proof. Certainly, Yu’s theorem was an inspiration for our theorem. The other source of inspiration was our recent joint work with Heath Emerson [9] in which we construct

finitely summable Fredholm modules for the C^* -crossed product $C(\partial\Gamma) \rtimes \Gamma$; the spiritual connection is the loose idea that the action of a hyperbolic group on its boundary is summable above the Hausdorff dimension of the boundary.

The proof of our theorem is neither shorter nor simpler than the proof of Yu's theorem, but it does have conceptual advantages. In the first part of the paper, we put forth the principle that Möbius actions give rise to affine isometric actions on L^p -spaces. This principle might encompass a number of situations, but here we focus on non-elementary hyperbolic groups. In this case, the Möbius philosophy yields the following:

- an explicit group-invariant measure on the square of the boundary, twin to the Bowen - Margulis measure encountered in the $\text{CAT}(-1)$ setting (compare with Furman [11, Prop.1]);
- a geometrically appealing cocycle, whose memorable form suggests that it should be interpreted as the *other* Busemann cocycle;
- an integrability exponent p which is related to a suitable interpretation of the Hausdorff dimension of the boundary, the *hyperbolic dimension* introduced by Mineyev [14].

See Sections 7 for precise statements. Differences aside, the two approaches – Yu's and ours – share a common technical point, and that is Mineyev's powerful “flower-averaging” procedure. For our purposes, the relevant upshot of this procedure is that it leads to visual metrics on the boundary with much better properties than the visual metrics coming from the word metric. These new visual metrics were constructed by Mineyev in [14].

Yet another result concerning proper isometric actions of hyperbolic groups on L^p -spaces is given by Bourdon in [5]. Bourdon's theorem, closer in spirit to Yu's original result, states the following: the linear isometric action of a hyperbolic group Γ on $\ell^p(\Gamma \times \cdots \times \Gamma)$ admits a proper cocycle for every $p > \text{Confdim } \partial\Gamma$. Here, too, the integrability exponent p is related to a suitable interpretation of the Hausdorff dimension of the boundary – the Ahlfors regular, conformal dimension Confdim . Note that, unlike Theorem 1.1, the linear part of the action requires several copies of the regular representation. Bourdon's construction of proper isometric actions is achieved via ℓ^p -cohomology; in particular, it has the advantage of dispensing with Mineyev's technical procedure.

On the other hand, our construction of proper isometric actions has the following consequence in ℓ^p -cohomology:

Theorem 1.3. *Let Γ be a non-elementary hyperbolic group. Then Γ admits a proper affine isometric action on the first ℓ^p -cohomology group $H_{(p)}^1(\Gamma)$ for large enough p .*

This is an intriguing strengthening of the fact, due to Gromov, that $H_{(p)}^1(\Gamma)$ is non-zero for large enough p . To establish Theorem 1.3, we use a result of Bourdon and Pajot [6].

2. PRELIMINARIES

2.1. Isometric actions on L^p -spaces. By the Mazur - Ulam theorem, isometric group actions on real Banach spaces are affine. An isometric action of a discrete group Γ on a real L^p -space is obtained from two ingredients. Firstly, a measured space X on which Γ acts in a measure-preserving way; then $(g, F) \mapsto g.F$ is a linear isometric action of Γ on $L^p(X)$. Secondly, a cocycle for this linear isometric action, that is a map $c : \Gamma \rightarrow L^p(X)$ satisfying $c_{gh} = g.c_h + c_g$ for all $g, h \in \Gamma$; then $(g, F) \mapsto g.F + c_g$ is an affine isometric action of Γ on $L^p(X)$. The latter isometric action is proper if $\|c_g\|_{L^p(X)} \rightarrow \infty$ as $g \rightarrow \infty$ in Γ .

2.2. Notation. We write \preccurlyeq to mean inequality up to a positive multiplicative constant, and the corresponding equivalence is denoted \asymp . The constants involved in these relations often depend on a number of parameters, and we record this dependence as subscripts (e.g., $\preccurlyeq_{p,\varepsilon}$).

3. PROPER ISOMETRIC ACTIONS OF FREE GROUPS ON L^p -SPACES

In this section, we give a simple and self-contained proof of Theorem 1.2 in the case of free groups. The purpose of this discussion is to foreshadow two key points, developed later: the construction of affine isometric actions on L^p -spaces from Möbius actions (Section 5), and the properness of the cocycle in the case of hyperbolic groups (Section 7).

3.1. The boundary Ω . Let \mathbb{F}_n be the free group on $n \geq 2$ generators, and put $q = 2n - 1$. The Cayley graph of \mathbb{F}_n with respect to the standard generators is the $2n$ -valent tree, rooted at the identity element. The boundary of this tree, customarily denoted by Ω , is the set of all infinite rooted paths without backtracks. We endow Ω with a probability measure μ defined by the requirement that

$$\mu(\Omega_x) = \frac{q}{q+1} q^{-|x|}$$

for all non-identity elements $x \in \mathbb{F}_n$. Here Ω_x denotes the “boundary under x ”, that is, the boundary subset consisting of all those $\omega \in \Omega$ which start with x , and $|x|$ is the length of x .

The Poisson kernel is given by

$$P_g(\omega) = q^{-|g|+2(g,\omega)} \quad (g \in \mathbb{F}_n, \omega \in \Omega)$$

where (\cdot, \cdot) , the Gromov product based at the identity, measures the length of the longest shared path. The Poisson kernel P_g represents the Radon - Nikodym derivative $d(g_*\mu)/d\mu$, and it satisfies the cocycle relation $P_{gh} = P_g \cdot g.P_h$ for $g, h \in \mathbb{F}_n$. (For more details see, for instance, Figà-Talamanca - Picardello [10].)

3.2. An invariant measure on $\Omega \times \Omega$. The following key relation is easy to check:

$$(\dagger) \quad q^{-2(g\xi, g\omega)} = P_{g^{-1}}(\xi) P_{g^{-1}}(\omega) q^{-2(\xi, \omega)} \quad (g \in \mathbb{F}_n, \xi, \omega \in \Omega)$$

The main consequence of this relation is that a suitably weighted product measure on $\Omega \times \Omega$ is \mathbb{F}_n -invariant. Namely, let ν be the measure on $\Omega \times \Omega$ given by

$$d\nu = q^{2(\xi, \omega)} d\mu(\xi) d\mu(\omega).$$

The diagonal of $\Omega \times \Omega$ is $(\mu \times \mu)$ -negligible, since points of Ω are μ -negligible. Therefore ν is well-defined, and the diagonal of $\Omega \times \Omega$ is ν -negligible as well.

Note also that ν is infinite and σ -finite. Indeed, consider the countable partition of $\Omega \times \Omega \setminus \text{diag}$ given by the sets $K_n = \{(\xi, \omega) \in \Omega \times \Omega : (\xi, \omega) = n\}$, where $n \geq 0$. We claim that $\nu(K_n) < \infty$ for all $n \geq 0$, and $\sum_{n \geq 0} \nu(K_n) = \infty$. Let $n \geq 1$. For each $\omega \in \Omega$, $\mu(\{\xi \in \Omega : (\xi, \omega) = n\}) = \frac{q-1}{q+1} q^{-n}$. Then $(\mu \times \mu)(K_n) = \frac{q-1}{q+1} q^{-n}$, hence $\nu(K_n) = \frac{q-1}{q+1} q^n$. A similar argument shows $\nu(K_0) = \frac{q}{q+1}$.

Lemma 3.1. ν is invariant for the diagonal action of \mathbb{F}_n .

Proof. Let $F \in L^1(\Omega \times \Omega, \nu)$ and $g \in \mathbb{F}_n$. Then:

$$\begin{aligned} \int g.F d\nu &= \iint F(g^{-1}\xi, g^{-1}\omega) q^{2(\xi, \omega)} d\mu(\xi) d\mu(\omega) \\ &= \iint F(\xi, \omega) q^{2(g\xi, g\omega)} dg^*\mu(\xi) dg^*\mu(\omega) \\ &= \iint F(\xi, \omega) q^{2(g\xi, g\omega)} P_{g^{-1}}(\xi) P_{g^{-1}}(\omega) d\mu(\xi) d\mu(\omega) \\ &= \iint F(\xi, \omega) q^{2(\xi, \omega)} d\mu(\xi) d\mu(\omega) = \int F d\nu \end{aligned}$$

The above computation involves a change of variables, followed by an application of (\dagger) . \square

3.3. A cocycle for the action on $\Omega \times \Omega$. A natural cocycle for the action of \mathbb{F}_n on Ω is the logarithm of the Poisson kernel $g \mapsto \log P_g$, and a cocycle in two variables can be obtained by taking the difference of two such cocycles. More precisely, if

$$c_g(\xi, \omega) := \frac{1}{2}(\log P_g(\xi) - \log P_g(\omega)) = (g, \xi) - (g, \omega) \quad (g \in \mathbb{F}_n, \xi, \omega \in \Omega)$$

then $g \mapsto c_g$ a cocycle for the diagonal action of \mathbb{F}_n on $\Omega \times \Omega$. The next proposition shows that c is a proper cocycle for the linear isometric representation of \mathbb{F}_n on $L^p(\Omega \times \Omega, \nu)$, for each $p \geq 1$.

Proposition 3.2. *For each $p \geq 1$ we have $\|c_g\|_{L^p(\nu)} \asymp_p |g|^{1/p}$.*

Proof. Let $g \in \mathbb{F}_n$ and write

$$\|c_g\|_{L^p(\nu)}^p = \iint |(g, \xi) - (g, \omega)|^p q^{2(\xi, \omega)} d\mu(\xi) d\mu(\omega).$$

As ξ runs over Ω , the Gromov product (g, ξ) takes on the values $0, 1, \dots, |g|$. Thus

$$\begin{aligned} \|c_g\|_{L^p(\nu)}^p &= \sum_{i,j=0}^{|g|} \iint_{\substack{(g, \xi)=i \\ (g, \omega)=j}} |(g, \xi) - (g, \omega)|^p q^{2(\xi, \omega)} d\mu(\xi) d\mu(\omega) \\ &= \sum_{i,j=0}^{|g|} |i-j|^p q^{2 \min\{i,j\}} \mu(\{\xi : (g, \xi) = i\}) \mu(\{\omega : (g, \omega) = j\}). \end{aligned}$$

For $i \in \{0, 1, \dots, |g|\}$, we have $\mu(\{\xi : (g, \xi) = i\}) \asymp q^{-i}$. In fact,

$$\frac{q-1}{q+1} q^{-i} \leq \mu(\{\xi : (g, \xi) = i\}) \leq \frac{q}{q+1} q^{-i}$$

with equality on the left for all $i \neq 0, |g|$, and equality on the right at the endpoints $0, |g|$. Hence, letting

$$S_N := \sum_{i,j=0}^N |i-j|^p q^{2 \min\{i,j\}} q^{-i} q^{-j} = \sum_{i,j=0}^N |i-j|^p q^{-|i-j|}$$

we have $\|c_g\|_{L^p(\nu)}^p \asymp S_{|g|}$. Now $S_N \asymp_p N$: the recurrence

$$S_{N+1} = \sum_{i,j=0}^{N+1} |i-j|^p q^{-|i-j|} = S_N + 2 \sum_{i=0}^N (N+1-i)^p q^{-(N+1-i)} = S_N + 2 \sum_{i=1}^{N+1} i^p q^{-i}$$

gives $S_N + 2q^{-1} \leq S_{N+1} < S_N + 2(\sum_{i \geq 1} i^p q^{-i})$. We conclude that $\|c_g\|_{L^p(\nu)}^p \asymp_p |g|$. \square

There are, certainly, easier ways to produce proper isometric actions of free groups on L^p -spaces. (Here is one such action. Let \vec{X} be the directed Cayley graph of \mathbb{F}_n with respect to the standard generators. Let $\{g \rightarrow h\}$ denote the shortest oriented path in \vec{X} from $g \in \mathbb{F}_n$ to $h \in \mathbb{F}_n$. Then

$$g \mapsto c_g := \sum_{e \in \{1 \rightarrow g\}} e - \sum_{e \in \{g \rightarrow 1\}} e$$

defines a proper cocycle for the linear isometric action of \mathbb{F}_n on the ℓ^p -space of the edge-set of \vec{X} .) Our aim in what follows is to promote the boundary-based method presented in this section to general non-elementary hyperbolic groups.

4. INTERLUDE: MÖBIUS CALCULUS

In this section we discuss “derivatives” of Möbius maps. The facts established herein will be used in the next section to construct affine isometric actions of Möbius groups on L^p -spaces.

Throughout, we let (X, d) be a compact metric space without isolated points and we consider Möbius self-homeomorphisms of X . Recall, the *cross-ratio* of a quadruple of distinct points in X is defined by the formula

$$(z_1, z_2; z_3, z_4) = \frac{d(z_1, z_3) d(z_2, z_4)}{d(z_1, z_4) d(z_2, z_3)}.$$

A homeomorphism $g : X \rightarrow X$ is called a *Möbius homeomorphism* if g preserves the cross-ratios, i.e., $(gz_1, gz_2; gz_3, gz_4) = (z_1, z_2; z_3, z_4)$ for all quadruples of distinct points $z_1, z_2, z_3, z_4 \in X$.

Lemma 4.1. *Let g be a self-homeomorphism of X . Then g is Möbius if and only if there exists a positive continuous function on X , denoted $|g'|$, with the property that for all $x, y \in X$ we have*

$$(*) \quad d^2(gx, gy) = |g'|(x) |g'|(y) d^2(x, y).$$

Before we prove the lemma, let us observe that a continuous function $|g'|$ satisfying $(*)$ has, in particular, the property that

$$\lim_{y \rightarrow x} \frac{d(gx, gy)}{d(x, y)} = |g'|(x)$$

for all $x \in X$. This property justifies the notation, as well as the interpretation of $|g'|$ as the *metric derivative* of g . Following Sullivan [18, Sec.4], we also interpret the relation $(*)$ as a geometric mean-value property.

Proof of Lemma 4.1. We prove the forward implication. The converse is a trivial verification.

Assume that g is a Möbius homeomorphism. Let x, u, v be a triple of distinct points in X . For any fourth distinct point y we have

$$\frac{d(gx, gy)}{d(x, y)} \frac{d(gu, gv)}{d(u, v)} = \frac{d(gx, gu)}{d(x, u)} \frac{d(gy, gv)}{d(y, v)}$$

since g preserves the cross-ratios. When $y \rightarrow x$, we obtain

$$(1) \quad \lim_{y \rightarrow x} \frac{d(gx, gy)}{d(x, y)} = \frac{d(gx, gu)}{d(x, u)} \frac{d(gx, gv)}{d(x, v)} \frac{d(u, v)}{d(gu, gv)}.$$

Let $|g'|_{u,v}$ denote the expression on the right-hand side of (1), viewed as a function of x . Then $|g'|_{u,v}$ is a positive continuous function on $X \setminus \{u, v\}$. However, the left-hand side of (1) is independent of the choice of u, v . Thus, picking \tilde{u}, \tilde{v} distinct points in $X \setminus \{u, v\}$, we have that $|g'|_{u,v} = |g'|_{\tilde{u}, \tilde{v}}$ on $X \setminus \{u, \tilde{u}, v, \tilde{v}\}$. Defining $|g'|$ on X as $|g'|_{u,v}$ on $X \setminus \{u, v\}$, and $|g'|_{\tilde{u}, \tilde{v}}$ on $X \setminus \{\tilde{u}, \tilde{v}\}$, we obtain a positive continuous function.

Now let us prove $(*)$ for distinct $x, y \in X$. Let u, v be distinct points in $X \setminus \{x, y\}$, so that we can use the local formula $|g'|_{u,v}$ for $|g'|$. That $d^2(gx, gy) = |g'|_{u,v}(x) |g'|_{u,v}(y) d^2(x, y)$ can be readily checked by rearranging factors and using the g -invariance of the cross-ratios. \square

The next lemma shows that metric derivatives are more than just continuous.

Lemma 4.2. *Let g be a Möbius self-homeomorphism of X . Then $|g'|$ is Lipschitz.*

Proof. We show that $\sqrt{|g'|}$ is Lipschitz. This is equivalent to $|g'|$ being Lipschitz, since

$$2 \sqrt{\min |g'|} \left| \sqrt{|g'|}(x) - \sqrt{|g'|}(y) \right| \leq \left| |g'| (x) - |g'| (y) \right| \leq 2 \sqrt{\max |g'|} \left| \sqrt{|g'|}(x) - \sqrt{|g'|}(y) \right|.$$

Let $x, y \in X$. For each $\varepsilon \in (0, 1)$, let $z_\varepsilon \in X \setminus \{y\}$ be such that $d(x, z_\varepsilon) \geq (1 - \varepsilon) \text{diam } X$. By the geometric mean-value property (*), we have

$$\sqrt{|g'|}(x) - \sqrt{|g'|}(y) = \frac{1}{\sqrt{|g'|}(z_\varepsilon)} \left(\frac{d(gx, gz_\varepsilon)}{d(x, z_\varepsilon)} - \frac{d(gy, gz_\varepsilon)}{d(y, z_\varepsilon)} \right).$$

Using the fact that g is $(\max |g'|)$ -Lipschitz, we estimate

$$\begin{aligned} \frac{d(gx, gz_\varepsilon)}{d(x, z_\varepsilon)} - \frac{d(gy, gz_\varepsilon)}{d(y, z_\varepsilon)} &\leq \frac{d(gx, gy)}{d(x, z_\varepsilon)} + \frac{d(gy, gz_\varepsilon)}{d(x, z_\varepsilon)} - \frac{d(gy, gz_\varepsilon)}{d(y, z_\varepsilon)} \\ &\leq \frac{d(gx, gy)}{d(x, z_\varepsilon)} + \frac{d(gy, gz_\varepsilon)}{d(y, z_\varepsilon)} \frac{d(x, y)}{d(x, z_\varepsilon)} \\ &\leq 2 (\max |g'|) \frac{d(x, y)}{d(x, z_\varepsilon)} \leq \frac{2 (\max |g'|)}{(1 - \varepsilon) \text{diam } X} d(x, y). \end{aligned}$$

Then

$$\sqrt{|g'|}(x) - \sqrt{|g'|}(y) \leq \frac{2 (\max |g'|)}{(1 - \varepsilon) (\text{diam } X) \sqrt{\min |g'|}} d(x, y),$$

and letting $\varepsilon \rightarrow 0$ we conclude that

$$(2) \quad \sqrt{|g'|}(x) - \sqrt{|g'|}(y) \leq \frac{2 (\max |g'|)}{(\text{diam } X) \sqrt{\min |g'|}} d(x, y).$$

So $\sqrt{|g'|}$ is, indeed, Lipschitz. \square

We now take a measure-theoretic turn. Recall that the Hausdorff measure of dimension $D \geq 0$ on X is defined by the formula

$$\mu_D(A) = \lim_{\delta \rightarrow 0} \left(\inf \left\{ \sum (\text{diam } U_i)^D : (U_i) \text{ } \delta\text{-cover of } A \right\} \right) \quad (A \subseteq X).$$

This is a Borel measure which is interesting for a single D only, the Hausdorff dimension of X .

The following lemma is a variation on a basic observation of Sullivan (cf. [18, p.174]).

Lemma 4.3. *Assume that X has finite, non-zero Hausdorff dimension D , and let g be a Möbius self-homeomorphism of X . Then $|g'|^D$ represents the Radon - Nikodym derivative $dg^* \mu_D / d\mu_D$.*

Proof. Fix a measurable subset $S \subseteq X$. We want to show that

$$(3) \quad \mu_D(gS) = \int_S |g'|^D d\mu_D.$$

From the geometric mean-value property (*), we get that

$$\left(\inf_U |g'| \right) \text{diam } U \leq \text{diam } gU \leq \left(\sup_U |g'| \right) \text{diam } U$$

for every $U \subseteq X$. It follows that, for every measurable $T \subseteq X$, we have

$$(4) \quad \left(\inf_T |g'| \right)^D \mu_D(T) \leq \mu_D(gT) \leq \left(\sup_T |g'| \right)^D \mu_D(T).$$

Now let $\varepsilon > 0$. Let also $\eta > 0$ be such that $||g'| (x) - |g'| (y)| < \varepsilon \min |g'|$ whenever $d(x, y) < \eta$. Hence, if T is a measurable η -set - in the sense that $\text{diam } T < \eta$ - then $\sup_T |g'| \leq (1 + \varepsilon) \inf_T |g'|$, which in turn yields

$$(1 + \varepsilon)^{-D} \left(\sup_T |g'| \right)^D \mu_D(T) \leq \int_T |g'|^D d\mu_D \leq (1 + \varepsilon)^D \left(\inf_T |g'| \right)^D \mu_D(T).$$

Thus, in light of (4), we get that for every measurable η -set T the following holds:

$$(5) \quad (1 + \varepsilon)^{-D} \mu_D(gT) \leq \int_T |g'|^D d\mu_D \leq (1 + \varepsilon)^D \mu_D(gT)$$

The measurable set S we started with may be partitioned into a finite number of measurable η -subsets. (Indeed, pick a finite cover of X by open η -subsets. This cover gives rise to a finite partition of X into Borel η -subsets, which partition can be used on any measurable subset of X .) Applying (5) to these η -pieces of S , and then adding up, we get

$$(1 + \varepsilon)^{-D} \mu_D(gS) \leq \int_S |g'|^D d\mu_D \leq (1 + \varepsilon)^D \mu_D(gS).$$

Since ε is arbitrary, we conclude that (3) holds. \square

5. ISOMETRIC ACTIONS OF MÖBIUS GROUPS ON L^p -SPACES

As before, (X, d) is a compact metric space without isolated points. Throughout this section we also assume that the Hausdorff dimension of X , denoted D , is finite and non-zero. The goal is to construct an affine isometric action of $\text{Möb}(X)$, the group of Möbius self-homeomorphisms of X , on $L^p(X \times X)$.

5.1. A Möbius-invariant measure on $X \times X$. The relevant measure on $X \times X$ is given by a suitable weighting of the product measure $\mu_D \times \mu_D$, where μ_D is the D -dimensional Hausdorff measure on X .

Lemma 5.1. *Assume $\mu_D(X) < \infty$. Then*

$$d\nu(x, y) = d^{-2D}(x, y) d\mu_D(x) d\mu_D(y)$$

defines a σ -finite Borel measure on $X \times X$. The diagonal of $X \times X$ is ν -negligible, and on the locally compact and σ -compact space $X \times X \setminus \text{diag}$, the measure ν is a Radon measure.

Proof. By the separability of X , the product measure $\mu_D \times \mu_D$ is a Borel measure on $X \times X$. Points are negligible for a Hausdorff measure of positive dimension, so the diagonal of $X \times X$ is $(\mu_D \times \mu_D)$ -negligible. Thus ν is well-defined, and the diagonal of $X \times X$ is ν -negligible. Since ν is obtained by weighting a Borel measure, namely $\mu_D \times \mu_D$, by a Borel map, namely $d^{-2D}(\cdot, \cdot)$, it follows that ν is Borel on $X \times X$. The σ -finiteness of ν , as well as the σ -compactness of the “slashed square” $X \times X \setminus \text{diag}$, follow by writing

$$X \times X \setminus \text{diag} = \bigcup_{n \geq 1} \left\{ (x, y) : d(x, y) \geq \frac{1}{n} \right\}.$$

When restricted to $X \times X \setminus \text{diag}$, ν is a Borel measure which is finite on compact subsets. The regularity of ν is automatic: it follows from [17, Thm.2.18] that, on a locally compact and σ -compact metric space, a Borel measure which is finite on compact sets is a Radon measure. \square

The key property of ν is its Möbius invariance.

Lemma 5.2. *Assume $\mu_D(X) < \infty$. Then ν is invariant for the diagonal action of $\text{Möb}(X)$.*

Proof. By the σ -finiteness of ν , it suffices to show that sets of finite measure are invariant. This is shown as in the proof of Lemma 3.1. For $F \in L^1(X \times X, \nu)$ and $g \in \text{Möb}(X)$ we have:

$$\begin{aligned} \int g.F d\nu &= \iint F(g^{-1}x, g^{-1}y) d^{-2D}(x, y) d\mu_D(x) d\mu_D(y) \\ &= \iint F(x, y) d^{-2D}(gx, gy) dg^* \mu_D(x) dg^* \mu_D(y) \\ &= \iint F(x, y) d^{-2D}(gx, gy) |g'|^D(x) |g'|^D(y) d\mu_D(x) d\mu_D(y) \\ &= \iint F(x, y) d^{-2D}(x, y) d\mu_D(x) d\mu_D(y) = \int F d\nu \end{aligned}$$

The second equality is a change of variables, the third relies on Lemma 4.3, and the fourth is due to the geometric mean-value property (*). \square

5.2. A cocycle for the Möbius action on $X \times X$. At this point, we have a linear isometric action of $\text{Möb}(X)$ on $L^p(X \times X, \nu)$ for each $p \geq 1$. We need a cocycle in order to get an affine isometric action, and this arises as follows. The metric derivatives satisfy the chain rule

$$(6) \quad |(gh)'|(x) = |g'|(|hx|) |h'|(|x|)$$

for all $g, h \in \text{Möb}(X)$ and $x \in X$. In other words, $g \mapsto \log |(g^{-1})'|$ is a cocycle for the action of $\text{Möb}(X)$ on X . Hence

$$g \mapsto c_g(x, y) := \log |(g^{-1})'|(|x|) - \log |(g^{-1})'|(|y|)$$

defines a cocycle for the diagonal action of $\text{Möb}(X)$ on $X \times X$. Now $g \mapsto c_g$ is a cocycle for the linear action of $\text{Möb}(X)$ on $L^p(X \times X, \nu)$ if and only if each c_g is in $L^p(X \times X, \nu)$. This turns out to be the case for all p greater than D , the Hausdorff dimension of X , as soon as we require μ_D to be *Ahlfors regular*. This means that the measure of balls, viewed as a function of the radius $r \in [0, \text{diam } X]$, satisfies

$$\mu_D(r\text{-ball}) \asymp r^D.$$

In particular, if μ_D is Ahlfors regular then $\mu_D(X) < \infty$.

Lemma 5.3. *Assume that μ_D is Ahlfors regular. Then $d \in L^{D, \infty}(X \times X, \nu) \setminus L^D(X \times X, \nu)$. Consequently, $d \in L^p(X \times X, \nu)$ for each $p > D$, and ν is an infinite measure on $X \times X$.*

Proof. To show that d is in the weak L^D -space $L^{D, \infty}(X \times X, \nu)$, we have to check that

$$\nu(\{(x, y) : d(x, y) > t\}) \lesssim t^{-D}$$

as t runs over positive reals. To that end, it suffices to show that for each fixed $y \in X$ we have

$$\int_{d(x, y) > t} d^{-2D}(x, y) \, d\mu_D(x) \lesssim t^{-D}$$

independent of y . And indeed:

$$\begin{aligned} \int_{d(x, y) > t} d^{-2D}(x, y) \, d\mu_D(x) &= \int_0^{t^{-2D}} \mu_D(\{x : d^{-2D}(x, y) > s\}) \, ds \\ &= \int_{(\text{diam } X)^{-2D}}^{t^{-2D}} \mu_D(\{x : d(x, y) < s^{-\frac{1}{2D}}\}) \, ds \\ &\leq \int_{(\text{diam } X)^{-2D}}^{t^{-2D}} C s^{-\frac{1}{2}} \, ds \leq (2C) t^{-D} \end{aligned}$$

The inequality in the last line uses the upper polynomial bound on the measure of balls. Next, we show that d is not in the subspace $L^D(X \times X, \nu) \subseteq L^{D, \infty}(X \times X, \nu)$, and here we use the lower polynomial bound on the measure of balls. For each $y \in X$ we have:

$$\begin{aligned} \int d^{-D}(x, y) \, d\mu_D(x) &= \int_0^\infty \mu_D(\{x : d^{-D}(x, y) > s\}) \, ds \\ &= \int_{(\text{diam } X)^{-D}}^\infty \mu_D(\{x : d(x, y) < s^{-\frac{1}{D}}\}) \, ds \\ &\geq \int_{(\text{diam } X)^{-D}}^\infty c s^{-1} \, ds = \infty \end{aligned}$$

Integrating with respect to y , we get $\|d\|_{L^D(\nu)} = \infty$.

The second part of the lemma follows by using the boundedness of d . As d is in weak $L^D(\nu)$ and in $L^\infty(\nu)$, interpolation yields that d is in $L^p(\nu)$ for each $p > D$. Finally, d not in $L^D(\nu)$ implies in particular that ν is infinite. \square

On the other hand, we may bound the cocycle by the metric as follows:

Lemma 5.4. *We have*

$$|c_g| \preceq \frac{\max |g'|}{\min |g'|} d.$$

Proof. Let $g \in \text{Möb}(X)$. The fact that $|\log a - \log b| \leq |a - b|/m$ whenever $a, b \geq m > 0$, and the Lipschitz estimate (2), yield

$$\begin{aligned} \left| \log |g'| (x) - \log |g'| (y) \right| &= 2 \left| \log \sqrt{|g'|} (x) - \log \sqrt{|g'|} (y) \right| \\ &\leq \frac{2}{\sqrt{\min |g'|}} \left| \sqrt{|g'|} (x) - \sqrt{|g'|} (y) \right| \leq \frac{4}{\text{diam } X} \frac{\max |g'|}{\min |g'|} d(x, y). \end{aligned}$$

The chain rule (6) implies that $|(g^{-1})'| = 1/g \cdot |g'|$, so $\max |(g^{-1})'| / \min |(g^{-1})'| = \max |g'| / \min |g'|$. Thus, the Lipschitz estimate for $\log |g'|$ becomes

$$|c_g(x, y)| \leq \frac{4}{\text{diam } X} \frac{\max |g'|}{\min |g'|} d(x, y),$$

as desired. \square

Lemma 5.3 and Lemma 5.4 imply that our cocycle c takes values in $L^p(X \times X, \nu)$ for each $p > D$. Summarizing, we have proved the following:

Proposition 5.5. *Assume that μ_D is Ahlfors regular, and let $p \geq 1$. Then, for each $p > D$,*

$$g \mapsto c_g(x, y) = \log |(g^{-1})'| (x) - \log |(g^{-1})'| (y)$$

is a cocycle for the linear isometric action of $\text{Möb}(X)$ on $L^p(X \times X, \nu)$.

5.3. A topological perspective. The groups we are ultimately interested in – namely, hyperbolic groups – are discrete. Proposition 5.5, in which we are treating the Möbius group $\text{Möb}(X)$ as a discrete group, suffices for our purposes. But the Möbius context we have developed so far has a topological layer as well, and we will look at it before moving on to the case of hyperbolic groups.

Recall that the natural topology on the space $C(Z, Y)$ of continuous maps between a compact space Z and a metric space Y is the compact-open topology or, equivalently, the topology of uniform convergence. This topology is induced by the metric

$$\text{dist}(f_1, f_2) = \sup_{z \in Z} d_Y(f_1 z, f_2 z).$$

For a compact metric space Z , the group of self-homeomorphisms $\text{Homeo}(Z)$ is a topological group under the topology of uniform convergence. In general, $\text{Homeo}(Z)$ is not locally compact.

Proposition 5.6. *Endow the Möbius group of X with the topology of uniform convergence, and let $p \geq 1$. Then:*

- i) *the metric differentiation map $\text{Möb}(X) \rightarrow C(X, \mathbb{R})$, given by $g \mapsto |g'|$ is continuous;*
- ii) *$\text{Möb}(X)$ is a locally compact and σ -compact topological group, containing $\text{Isom}(X)$ as a compact subgroup;*
- iii) *for each $p > D$, the affine isometric action of $\text{Möb}(X)$ on $L^p(X \times X, \nu)$, given by $(g, F) \mapsto g.F + c_g$, is continuous.*

Proof. We start by showing that

$$(7) \quad \text{dist}(|g'|, 1) \leq C^{-1} \text{dist}(g, \text{id}), \text{ provided that } \text{dist}(g, \text{id}) \leq C$$

for some $C > 0$ depending on X only.

There is $\kappa > 0$ such that no two open balls of radius κ cover X ; the easy proof, by contradiction, is left to the reader. Now assume that $D(g) := \text{dist}(g, \text{id}) \leq \kappa/10$. Fix $x \in X$. By the defining property of κ , there are $u, v \in X$ such that $d(x, u) \geq \kappa$, $d(u, v) \geq \kappa$, $d(x, v) \geq \kappa$. Recall from the proof of Lemma 4.1 the following local formula:

$$|g'| (x) = \frac{d(gx, gu)}{d(x, u)} \frac{d(gx, gv)}{d(x, v)} \frac{d(u, v)}{d(gu, gv)}$$

We have $d(x, u) - 2D(g) \leq d(gx, gu) \leq d(x, u) + 2D(g)$. As $d(x, u) \geq \kappa$, we obtain

$$1 - 2\kappa^{-1}D(g) \leq \frac{d(gx, gu)}{d(x, u)} \leq 1 + 2\kappa^{-1}D(g).$$

The same bounds are valid for x and v , and for u and v , instead of x and u . Therefore

$$\frac{(1 - 2\kappa^{-1}D(g))^2}{1 + 2\kappa^{-1}D(g)} - 1 \leq |g'| (x) - 1 \leq \frac{(1 + 2\kappa^{-1}D(g))^2}{1 - 2\kappa^{-1}D(g)} - 1.$$

The lower bound is greater than $-6\kappa^{-1}D(g)$, whereas the upper bound is at most $8\kappa^{-1}D(g)$. The claim (7) is thus proved, with $C := \kappa/10$.

i) By (7), the metric differentiation map is continuous at the identity element of $\text{Möb}(X)$. The continuity on $\text{Möb}(X)$ follows, since $\text{dist}(|g'|, |h'|) \leq (\max |h'|) \text{dist}(|(gh^{-1})'|, 1)$ for all $g, h \in \text{Möb}(X)$ by using the chain rule (6).

ii) It is clear that $\text{Möb}(X)$ is a topological group, and a closed subgroup of $\text{Homeo}(X)$. For each $R \geq 1$ the subset $\{g \in \text{Möb}(X) : \max |g'| \leq R\}$ is closed and equicontinuous, hence compact by the Arzelà - Ascoli theorem. On the one hand, it follows that $\text{Möb}(X)$ is σ -compact. On the other hand, (7) implies that the closed ball of radius C around the identity is contained in $\{g \in \text{Möb}(X) : \max |g'| \leq 2\}$. Since the latter is compact, the former is a compact neighborhood of the identity, and we conclude that $\text{Möb}(X)$ is locally compact. In what concerns the compactness of the isometry group, note that $\text{Isom}(X) = \{g \in \text{Möb}(X) : \max |g'| \leq 1\}$.

iii) By definition, an affine isometric action of a topological group on a Banach space is continuous if the linear part of the action is strongly (that is, pointwise) continuous, and the cocycle is (norm) continuous.

Fix $p > D$. First, we show that the linear isometric action $(g, F) \mapsto g.F$ of $\text{Möb}(X)$ on $L^p(X \times X, \nu)$ is strongly continuous. To that end, it suffices to check that for a dense set of functions F in $L^p(X \times X, \nu)$ we have $g.F \rightarrow F$ in $L^p(X \times X, \nu)$ whenever $g \rightarrow \text{id}$ in $\text{Möb}(X)$. Recall from Lemma 5.1 that ν is a Radon measure on the locally compact space $X^{2-} := X \times X \setminus \text{diag}$. Hence $C_c(X^{2-})$, the subspace of compactly-supported continuous functions on X^{2-} , is dense in $L^p(X^{2-}, \nu) = L^p(X \times X, \nu)$. Let $F \in C_c(X^{2-})$. If $g \rightarrow \text{id}$ in $\text{Möb}(X)$ then $g.F \rightarrow F$ uniformly. Since $\|g.F - F\|_{L^p(X^{2-}, \nu)} \leq \|g.F - F\|_\infty \nu(\text{supp } F)$, where $\nu(\text{supp } F) < \infty$, we conclude that $g.F \rightarrow F$ in $L^p(X \times X, \nu)$.

To show that the cocycle $g \mapsto c_g$ is continuous, it suffices to check continuity at the identity element of $\text{Möb}(X)$; the cocycle rule will then imply continuity at every element of $\text{Möb}(X)$. Let $r \in (D, p)$. Then

$$\|c_g\|_{L^p(\nu)}^p \leq \|c_g\|_\infty^{p-r} \|c_g\|_{L^r(\nu)}^r.$$

We have

$$\|c_g\|_\infty = \log \frac{\max |g'|}{\min |g'|}, \quad \|c_g\|_{L^r(\nu)} \preccurlyeq_r \frac{\max |g'|}{\min |g'|},$$

the latter by Lemma 5.4. Therefore

$$\|c_g\|_{L^p(\nu)}^p \preccurlyeq_r \left(\log \frac{\max |g'|}{\min |g'|} \right)^{p-r} \left(\frac{\max |g'|}{\min |g'|} \right)^r,$$

from which it follows that $\|c_g\|_{L^p(\nu)} \rightarrow 0$ as $g \rightarrow \text{id}$ in $\text{Möb}(X)$. \square

6. INTERLUDE: VISUAL METRICS ON BOUNDARIES OF HYPERBOLIC GROUPS

The next goal is to apply the construction of Section 5 to the action of a non-elementary hyperbolic group on its boundary. In order to do so, we need a metric on the boundary such that the group action is by Möbius maps, and such that the corresponding Hausdorff measure is Ahlfors regular.

6.1. Visual metrics induced by the word metric. Let Γ be a non-elementary hyperbolic group, and consider the Cayley graph of Γ with respect to a finite generating set. Throughout, we choose the identity element as the basepoint.

In the beginning there is the word metric, and the word metric is geodesic. Let $\partial\Gamma$ denote the Gromov boundary of the Cayley graph. Topologically, $\partial\Gamma$ is a canonical compact space without isolated points on which Γ acts by homeomorphisms. The metric structure on $\partial\Gamma$ is, however, more subtle. The classical approach runs as follows. First, the Gromov product (\cdot, \cdot) on $\Gamma \times \Gamma$ is extended to $\bar{\Gamma} \times \bar{\Gamma}$, where $\bar{\Gamma} = \Gamma \cup \partial\Gamma$ is the boundary compactification. Such an extension involves a somewhat ad-hoc choice and, a priori, it is neither unique nor continuous. Next, the expression $\exp(-\varepsilon(\cdot, \cdot))$ is turned into a compatible metric on the boundary. More precisely, for each sufficiently small parameter $\varepsilon > 0$ there is a *visual metric* $d_\varepsilon \asymp \exp(-\varepsilon(\cdot, \cdot))$ which agrees with the canonical topology on $\partial\Gamma$. (See, for instance, Bourdon [4] and Kapovich - Benakli [12] for more details.) For a visual metric d_ε , the geometric mean-value property $(*)$ holds up to multiplicative constants depending on the visual parameter ε and the hyperbolicity constant δ . Clearly, starting with a “quasi-definition” of the extended Gromov product snowballs into a quasified metric structure on $\partial\Gamma$.

The Patterson - Sullivan theory for non-elementary hyperbolic groups, due to Coornaert [8], describes the Hausdorff dimensions and the Hausdorff measures associated to visual metrics on the boundary. Let

$$e(\Gamma) = \limsup_{n \rightarrow \infty} \frac{\log \#\{g \in \Gamma : l(g) \leq n\}}{n}$$

be the growth exponent of Γ , where l denotes the word length. Then $0 < e(\Gamma) < \infty$, and the following holds:

Proposition 6.1 ([8]). *Equip $\partial\Gamma$ with a visual metric d_ε . If D_ε denotes the Hausdorff dimension of $\partial\Gamma$, then $D_\varepsilon = e(\Gamma)/\varepsilon$ and the D_ε -dimensional Hausdorff measure is Ahlfors regular.*

6.2. Visual metrics induced by Mineyev’s hat metric. In the $\text{CAT}(-1)$ setting, the metric structure on the boundary is better behaved. Namely, let X be a proper $\text{CAT}(-1)$ space with a fixed basepoint $o \in X$. Then the Gromov product $(\cdot, \cdot)_o$ on $X \times X$ extends continuously to $\bar{X} \times \bar{X}$, and for each $\varepsilon \in (0, 1]$ the expression $\exp(-\varepsilon(\cdot, \cdot)_o)$ is a compatible metric on the boundary ∂X (cf. Bourdon [4]). If we equip ∂X with a visual metric $\exp(-\varepsilon(\cdot, \cdot)_o)$, then the action of $\text{Isom}(X)$ on ∂X is Möbius. Furthermore, the Möbius group of ∂X does not depend on the visual parameter ε .

Mineyev showed in [14] that properties similar to the ones in the $\text{CAT}(-1)$ setting can be achieved on boundaries of hyperbolic groups. The main technical point is the replacement of the word metric on the group by a new metric, herein called the *hat metric*. First introduced by Mineyev and Yu in [15], the hat metric is the key geometric ingredient in their proof of the Baum-Connes conjecture for hyperbolic groups.

As before, let Γ be a non-elementary hyperbolic group. Let X be the Cayley graph of Γ with respect to a finite generating set, and endow X with the path metric d . The following result collects the properties of the hat metric which are relevant for this paper.

Proposition 6.2 ([14, 13]). *There is a metric \hat{d} on X having the following properties:*

- i) \hat{d} is Γ -invariant, quasi-isometric to d , and roughly geodesic;
- ii) for each $o \in X$, the Gromov product $\langle \cdot, \cdot \rangle_o$ with respect to \hat{d} on X extends to a continuous map $\langle \cdot, \cdot \rangle_o : \overline{X} \times \overline{X} \rightarrow [0, \infty]$, where $\langle \xi, \omega \rangle_o = \infty$ if and only if $\xi = \omega \in \partial X$;
- iii) (normalization) for each $\varepsilon \in (0, 1]$, $\exp(-\varepsilon \langle \cdot, \cdot \rangle_o)$ is a metric on ∂X for all $o \in X$.

Thus, by passing from the path metric d to the hat metric \hat{d} , we get a Gromov product which behaves just like the one in the CAT(-1) setting. The metrics on the boundary $\partial X = \partial \Gamma$ are then simple and explicit, and they also have sharp properties. In the direction that concerns us, we have that the action on the boundary is Möbius – whereas classically the action is quasi-Möbius only. Recall, the identity element of Γ is our chosen basepoint.

Corollary 6.3 ([14]). *Equip $\partial \Gamma$ with a visual metric $\hat{d}_\varepsilon = \exp(-\varepsilon \langle \cdot, \cdot \rangle)$ defined by the Gromov product with respect to \hat{d} . Then Γ acts on $\partial \Gamma$ by Möbius homeomorphisms.*

Note also that the Möbius group of $\partial \Gamma$ is independent of the choice of visual parameter ε .

Proof. We verify that the geometric mean-value property (*) holds for each $g \in \Gamma$. A direct calculation shows that

$$2\langle gx, gw \rangle - 2\langle x, w \rangle = (\hat{l}(g^{-1}) - 2\langle g^{-1}, x \rangle) + (\hat{l}(g^{-1}) - 2\langle g^{-1}, w \rangle) \quad (x, w \in \Gamma),$$

where $\hat{l}(g) := \hat{d}(1, g)$ is the hat length of g . Letting $x \rightarrow \xi$ and $w \rightarrow \omega$, we get

$$(8) \quad \hat{d}_\varepsilon^2(g\xi, g\omega) = \exp\left(\varepsilon(2\langle g^{-1}, \xi \rangle - \hat{l}(g^{-1}))\right) \exp\left(\varepsilon(2\langle g^{-1}, \omega \rangle - \hat{l}(g^{-1}))\right) \hat{d}_\varepsilon^2(\xi, \omega)$$

for all $\xi, \omega \in \partial \Gamma$. □

Minyev's hat metric straightens the outside while wrinkling the inside. Indeed, the previous corollary witnesses the fact that exchanging the path metric d for the hat metric \hat{d} improves the metric structure on the boundary. But there is a price to be paid within the space X : while the path metric d is geodesic, the hat metric \hat{d} is only roughly geodesic. Recall, to say that \hat{d} is *roughly geodesic* (+geodesic, in the language of [14]) is to mean the following: there is a spatial constant $C \geq 0$ with the property that, for any points $x, y \in X$, there is a (not necessarily continuous) map $\gamma : [a, b] \rightarrow \mathbb{R}$ such that $\gamma(a) = x$, $\gamma(b) = y$, and γ is a C -rough isometry, i.e.,

$$|s - t| - C \leq \hat{d}(\gamma(s), \gamma(t)) \leq |s - t| + C$$

for all $s, t \in [a, b]$. Hyperbolicity is a quasi-isometry invariant for roughly geodesic spaces, so \hat{d} is a hyperbolic metric on X .

Recent work of Blachère - Haïssinsky - Mathieu shows that we still have Ahlfors regularity for the visual metrics coming from the hat metric. More precisely, [3, Thm.2.3] leads to the following:

Proposition 6.4. *Equip $\partial \Gamma$ with a visual metric $\hat{d}_\varepsilon = \exp(-\varepsilon \langle \cdot, \cdot \rangle)$ defined by the Gromov product with respect to \hat{d} . If \hat{D}_ε denotes the Hausdorff dimension of $\partial \Gamma$, then $\hat{D}_\varepsilon = \hat{e}(\Gamma)/\varepsilon$ and the \hat{D}_ε -dimensional Hausdorff measure is Ahlfors regular.*

Here

$$\hat{e}(\Gamma) = \limsup_{n \rightarrow \infty} \frac{\log \#\{g \in \Gamma : \hat{l}(g) \leq n\}}{n}$$

is the growth exponent of Γ with respect to the hat length \hat{l} .

To conclude, Corollary 6.3 and Proposition 6.4 fulfill the two desiderata announced at the beginning of the section.

Remark 6.5. In [3], Blachère, Haïssinsky, and Mathieu are concerned with another interesting metric on hyperbolic groups, the Green metric coming from a random walk. The Green metric is a more natural, and much easier to construct, replacement of the word metric than the hat metric. It is unclear whether these two exotic metrics on hyperbolic groups, the Green metric and the hat metric, are somehow related. It is also unclear whether the Green metric leads to a metric structure on the boundary as nice as the one coming from the hat metric.

7. PROPER ISOMETRIC ACTIONS OF HYPERBOLIC GROUPS ON L^p -SPACES

Again, Γ is a non-elementary hyperbolic group. Equip $\partial\Gamma$ with a visual metric $\hat{d}_\varepsilon = \exp(-\varepsilon\langle \cdot, \cdot \rangle)$ defined by the Gromov product with respect to the hat metric \hat{d} . Then, as explained in the previous section, the action of Γ on $\partial\Gamma$ falls under the framework developed in Section 5. Let us see what we obtain.

7.1. A Γ -invariant measure on $\partial\Gamma \times \partial\Gamma$. While the Hausdorff dimension of $\partial\Gamma$ depends on ε , the Hausdorff measure on $\partial\Gamma$ does not, and we simply denote it by μ . The Möbius-invariant measure ν on $\partial\Gamma \times \partial\Gamma$ is also independent of the visual parameter, and it takes the form

$$(9) \quad d\nu(\xi, \omega) = \exp(2\hat{e}(\Gamma)(\xi, \omega)) d\mu(\xi) d\mu(\omega).$$

In particular, ν is Γ -invariant. Since the diagonal of $\partial\Gamma \times \partial\Gamma$ is ν -negligible, we may also view ν as a Γ -invariant Radon measure on $\partial^2\Gamma := \partial\Gamma \times \partial\Gamma \setminus \text{diag}$.

The measure ν is a generalization of the Bowen - Margulis measure from the $\text{CAT}(-1)$ setting. Namely, consider the special case when Γ acts geometrically on a proper $\text{CAT}(-1)$ space X . If we forgo the visual metrics coming from the hat metric on the Cayley graph of Γ , and we use instead the visual metrics coming from within X , then the Möbius-invariant measure ν we obtain is the so-called *Bowen - Margulis measure*. This has the form

$$d\nu_{\text{BM}}(\xi, \omega) = \frac{d\mu_o(\xi) d\mu_o(\omega)}{d_{\varepsilon,o}(\xi, \omega)^{2D_\varepsilon}} = \exp(2e(\Gamma)(\xi, \omega)_o) d\mu_o(\xi) d\mu_o(\omega),$$

independent of the basepoint $o \in X$. Here D_ε and μ_o denote the Hausdorff dimension, respectively the Hausdorff measure, with respect to the visual metric $d_{\varepsilon,o} = \exp(-\varepsilon\langle \cdot, \cdot \rangle_o)$.

The existence of a Γ -invariant measure on $\partial\Gamma \times \partial\Gamma$ analogous to the Bowen - Margulis measure was first established by Furman in [11, Prop.1] using a cohomological argument. Roughly speaking, Furman constructs the measure in the loose measurable sense whereas our ν is constructed in the sharp metric category. This grants ν some advantages: it is more explicit, it is much closer to the Bowen - Margulis measure, and it has the sharp properties needed for the construction of a proper isometric action on $L^p(\partial\Gamma \times \partial\Gamma)$.

7.2. A cocycle for the Γ -action on $\partial\Gamma \times \partial\Gamma$. By (8), the metric derivative of $g \in \Gamma$ is

$$(10) \quad |g'|^{(\varepsilon)}(\xi) = \exp\left(\varepsilon(2\langle g^{-1}, \xi \rangle - \hat{l}(g^{-1}))\right),$$

and the cocycle c becomes, up to a factor of 2ε ,

$$(11) \quad g \mapsto c_g(\xi, \omega) = \langle g, \xi \rangle - \langle g, \omega \rangle.$$

We may interpret this attractive cocycle as follows. For $g \in \Gamma$ and $\xi \in \partial\Gamma$, let $\hat{\beta}(g, \xi) := 2\langle g, \xi \rangle - \hat{l}(g)$. If we fix a boundary point ξ and we view $\hat{\beta}$ as a function on the group, then $\hat{\beta}(\cdot, \xi)$ is the *Busemann function* corresponding to ξ , and the difference map $(g, h) \mapsto \hat{\beta}(g, \xi) - \hat{\beta}(h, \xi)$ is the *Busemann cocycle* with respect to ξ . If we fix a group element g and we view $\hat{\beta}$ as a

function on the boundary, then the difference map $(\xi, \omega) \mapsto \hat{\beta}(g, \xi) - \hat{\beta}(g, \omega)$ is, up to a factor of 2, our cocycle c_g . We thus think of c as the *other Busemann cocycle*.

Let $p \geq 1$ be such that p is greater than \hat{D}_ε , the Hausdorff dimension of $(\partial\Gamma, \hat{d}_\varepsilon)$. By Proposition 5.5, c is a cocycle for the linear isometric action of Γ on $L^p(\partial\Gamma \times \partial\Gamma)$. The last fact we have to prove is the following:

Proposition 7.1. *The cocycle c is proper.*

Proof. Let $\hat{\delta}$ denote the hyperbolicity constant of \hat{d} . Put $K := e^{\hat{\delta}(\Gamma)}$. Then

$$\begin{aligned} \|c_g\|_{L^p(\nu)}^p &= \iint |\langle g, \xi \rangle - \langle g, \omega \rangle|^p K^{2\langle \xi, \omega \rangle} d\mu(\xi) d\mu(\omega) \\ &\geq K^{-2\hat{\delta}} \iint |\langle g, \xi \rangle - \langle g, \omega \rangle|^p K^{2\min\{\langle g, \xi \rangle, \langle g, \omega \rangle\}} d\mu(\xi) d\mu(\omega) \end{aligned}$$

by using the hyperbolic inequality $\langle \xi, \omega \rangle \geq \min\{\langle g, \xi \rangle, \langle g, \omega \rangle\} - \hat{\delta}$.

Let $M(g) := \max_{\xi \in \partial\Gamma} \langle g, \xi \rangle$. Observe that $M(g) \rightarrow \infty$ as $g \rightarrow \infty$ in Γ . Indeed, assuming the contrary means that there is $C > 0$ and sequence $g_n \rightarrow \infty$ such that $\langle g_n, \xi \rangle \leq C$ for all n and all $\xi \in \partial\Gamma$. Let $\omega \in \partial\Gamma$ be an accumulation point of (g_n) . Then $\langle \omega, \xi \rangle \leq C$ for all $\xi \in \partial\Gamma$. But this is a contradiction, since $\langle \omega, \omega \rangle = \infty$.

We claim that for all $0 \leq R \leq M(g) - \hat{\delta}$ we have $\mu(\{\xi \in \partial\Gamma : \langle g, \xi \rangle \geq R\}) \asymp_{\varepsilon, \hat{\delta}} K^{-R}$. To prove the claim, pick $\omega \in \partial\Gamma$ such that $\langle g, \omega \rangle = M(g) \geq R + \hat{\delta}$. Then

$$(12) \quad B(\omega, e^{-\varepsilon(R+\hat{\delta})}) \subseteq \{\xi \in \partial\Gamma : \langle g, \xi \rangle \geq R\} \subseteq B(\omega, e^{-\varepsilon(R-\hat{\delta})})$$

where the balls are closed, and taken with respect to the \hat{d}_ε metric. Indeed, let $\xi \in B(\omega, e^{-\varepsilon(R+\hat{\delta})})$; then $\langle \xi, \omega \rangle \geq R + \hat{\delta}$, hence $\langle g, \xi \rangle \geq \min\{\langle g, \omega \rangle, \langle \xi, \omega \rangle\} - \hat{\delta} \geq R$. This justifies the first inclusion. For the second one, let $\xi \in \partial\Gamma$ with $\langle g, \xi \rangle \geq R$; then $\langle \xi, \omega \rangle \geq \min\{\langle g, \xi \rangle, \langle g, \omega \rangle\} - \hat{\delta} \geq R - \hat{\delta}$, hence $\hat{d}_\varepsilon(\xi, \omega) \leq e^{-\varepsilon(R-\hat{\delta})}$. The claimed measure-theoretic estimate follows from (12) together with the Ahlfors regularity of μ .

From the above claim, it follows that there exists a constant $\rho = \rho(\varepsilon) > 0$ such that the measure of the set

$$A_k := \{\xi \in \partial\Gamma : (k+1)\rho > \langle g, \xi \rangle \geq k\rho\}$$

satisfies $\mu(A_k) \asymp_\varepsilon K^{-k\rho}$ for all $0 \leq k \leq N_\varepsilon(g)$, where $N_\varepsilon(g) := \lfloor (M(g) - \hat{\delta})/\rho \rfloor - 1$.

Splitting over the annular sets A_k , we have:

$$\|c_g\|_{L^p(\nu)}^p \geq_{\hat{\delta}} \sum_{0 \leq j < k \leq N_\varepsilon(g)} \iint_{\substack{\xi \in A_k \\ \omega \in A_j}} |\langle g, \xi \rangle - \langle g, \omega \rangle|^p K^{2\min\{\langle g, \xi \rangle, \langle g, \omega \rangle\}} d\mu(\xi) d\mu(\omega)$$

For $\xi \in A_k$ and $\omega \in A_j$ with $k > j$, we have $|\langle g, \xi \rangle - \langle g, \omega \rangle| = \langle g, \xi \rangle - \langle g, \omega \rangle \geq (k-j-1)\rho$ and $\min\{\langle g, \xi \rangle, \langle g, \omega \rangle\} = \langle g, \omega \rangle \geq j\rho$. Hence

$$\begin{aligned} \|c_g\|_{L^p(\nu)}^p &\geq_{\varepsilon, \hat{\delta}} \sum_{0 \leq j < k \leq N_\varepsilon(g)} (k-j-1)^p K^{2j\rho} K^{-k\rho} K^{-j\rho} \\ &= K^{-\rho} \sum_{0 \leq j \leq k \leq N_\varepsilon(g)-1} (k-j)^p K^{-(k-j)\rho} \geq K^{-2\rho} (N_\varepsilon(g) - 1), \end{aligned}$$

where the last inequality follows from a simple recurrence, as in the proof of Lemma 3.2. We are done, since $N_\varepsilon(g) \rightarrow \infty$ as $g \rightarrow \infty$ in Γ . \square

7.3. Hyperbolic dimension. The best integrability exponent p in sight is obtained by taking the visual parameter ε to be 1. Then we obtain a proper action of Γ on $L^p(\partial\Gamma \times \partial\Gamma)$ for every $p > \hat{e}(\Gamma)$; recall, $\hat{e}(\Gamma)$ is the growth exponent of Γ with respect to the hat metric \hat{d} . However, we can do slightly better. Consider the infimum of all Hausdorff dimensions of the boundary with respect to visual metrics $\exp(-\varepsilon\langle \cdot, \cdot \rangle)$ which come from some hat metric, i.e., a metric satisfying properties i), ii), and iii) of Theorem 6.2. This infimum is Mineyev's *hyperbolic dimension* (cf. [14, Section 10]). The notion of hyperbolic dimension is inspired by Pansu's conformal dimension, and one can see that the hyperbolic dimension is at least as large as the the Ahlfors regular, conformal dimension mentioned in the introduction.

Let $h(\Gamma)$ denote the hyperbolic dimension of a non-elementary hyperbolic group Γ . Then $h(\Gamma)$ is a non-negative real number equal to 0 if Γ is virtually free, and greater or equal to 1 otherwise [14, Thm.22 & Cor.23].

Example 7.2. Consider the rank-1 symmetric space \mathbb{H}_K^n , where $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $n \geq 2$, or $K = \mathbb{O}$ and $n = 2$, normalized so that the maximal sectional curvature is -1 . If Γ is a cocompact lattice in $\text{Isom}(\mathbb{H}_K^n)$ then, by a result of Pansu [16], we get that $h(\Gamma) = nk + k - 2$ where $k = \dim_{\mathbb{R}} K$.

With the notion of hyperbolic dimension at hand, our main result can be neatly stated as follows:

Theorem 7.3. *Let Γ be a non-elementary hyperbolic group, and let $p \geq 1$. Then Γ admits a proper affine isometric action on $L^p(\partial\Gamma \times \partial\Gamma)$ for all $p > h(\Gamma)$.*

In particular, Theorem 7.3 together with the Fisher - Margulis result that a group with property (T) is $L^{2+\varepsilon}$ -rigid (see [1, Thm.1.3]) imply the following consequence: if Γ is an infinite hyperbolic group with property (T), then $h(\Gamma) > 2$.

8. AN ℓ^p -COHOMOLOGICAL INTERPRETATION

8.1. Besov spaces. Let us return to the setup of Section 5, where we considered compact metric spaces whose Hausdorff measure is Ahlfors regular. Recall the notation: (X, d) is a compact metric space of Hausdorff dimension $D \in (0, \infty)$, and the D -dimensional Hausdorff measure is denoted μ_D . Let also $p \in [1, \infty)$.

Following Bourdon and Pajot [6], we consider the *Besov space* $B_p(X)$. The Besov seminorm of a measurable function $f : X \rightarrow \mathbb{R}$ is given by

$$\|f\|_{B_p} = \left(\iint |f(x) - f(y)|^p d^{-2D}(x, y) d\mu_D(x) d\mu_D(y) \right)^{\frac{1}{p}}.$$

Note that $\|f\|_{B_p} = 0$ if and only if f is a.e. constant. The Besov space $B_p(X)$ is the space of functions having finite Besov seminorm, modulo a.e. constant functions. Equipped with the induced Besov norm, $B_p(X)$ is a Banach space. There is an obvious isometric embedding

$$B_p(X) \longrightarrow L^p(X \times X, \nu), \quad [f] \longmapsto ((x, y) \mapsto f(x) - f(y))$$

where, we recall, $d\nu(x, y) = d^{-2D}(x, y) d\mu_D(x) d\mu_D(y)$. For $p > D$, we also have a continuous embedding $\text{Lip}(X) \hookrightarrow B_p(X)$; here $\text{Lip}(X)$ is the space of Lipschitz functions modulo constant functions, equipped with the induced Lipschitz norm. Indeed, $\|[f]\|_{B_p} \leq \|[f]\|_{\text{Lip}} \|d\|_{L^p(\nu)}$ for $[f] \in \text{Lip}(X)$, and $d \in L^p(X \times X, \nu)$ whenever $p > D$ by Lemma 5.3.

Now we see that the affine isometric action of the Möbius group $\text{Möb}(X)$ on the L^p -space $L^p(X \times X)$ is, in essence, an affine isometric action on the Besov space $B_p(X)$. More precisely, for each $p > D$, the map $g \mapsto \log |(g^{-1})'|$ is a (Lipschitz) cocycle for the linear isometric action of $\text{Möb}(X)$ on $B_p(X)$.

Moving to the context of hyperbolic groups, recall that the boundary $\partial\Gamma$ of a non-elementary hyperbolic group Γ is metrized by some visual metric $\hat{d}_\varepsilon = \exp(-\varepsilon\langle\cdot, \cdot\rangle)$. However, (9) shows that the Γ -invariant Besov norm on the boundary $\partial\Gamma$ is given by

$$\|f\|_{B_p} = \left(\iint |f(\xi) - f(\omega)|^p \exp(2\hat{e}(\Gamma)\langle\xi, \omega\rangle) d\mu(\xi) d\mu(\omega) \right)^{\frac{1}{p}},$$

independent of the choice of visual parameter ε . By (10), the cocycle $g \mapsto \log |(g^{-1})'|$ becomes $g \mapsto \hat{\beta}(g, \cdot) = 2\langle g, \cdot \rangle - \hat{l}(g)$. This too could be called the Busemann cocycle (compare the discussion in Section 7.2), and it is a Lipschitz cocycle for any choice of visual metric $\hat{d}_\varepsilon = \exp(-\varepsilon\langle\cdot, \cdot\rangle)$. When we work modulo constant functions, as we do in the Lipschitz space Lip and in the Besov spaces B_p , this simple Busemann cocycle simplifies further to $g \mapsto [\langle g, \cdot \rangle]$. We thus have:

Proposition 8.1. *Let Γ be a non-elementary hyperbolic group, and let $p \geq 1$. Then the linear isometric action of Γ on $B_p(\partial\Gamma)$ admits $g \mapsto [\langle g, \cdot \rangle]$ as a proper cocycle for all $p > \hat{e}(\Gamma)$.*

8.2. ℓ^p -cohomology in degree one. Let Γ be a finitely generated group, and consider the Cayley graph of Γ with respect to a finite generating set. For $p \in [1, \infty)$, we let

$$E_p(\Gamma) = \left\{ \phi : \Gamma \rightarrow \mathbb{R} : \|\phi\|_{E_p} = \left(\sum_{x \bullet \bullet y} |\phi(x) - \phi(y)|^p \right)^{\frac{1}{p}} < \infty \right\}$$

be the linear space of functions on Γ with p -summable edge differential. The *first ℓ^p -cohomology group* $H_{(p)}^1(\Gamma)$ is defined as the quotient of $E_p(\Gamma)$ by $\ell^p\Gamma + \mathbb{R}$.

The E_p -seminorms with respect to two Cayley graphs of Γ are comparable, meaning that the linear space $E_p(\Gamma)$ and the isomorphism type of the seminormed space $(E_p(\Gamma), \|\cdot\|_{E_p})$ are both independent of the choice of Cayley graph for Γ . Furthermore, the E_p -seminorm is Γ -invariant. All these features are inherited by $H_{(p)}^1(\Gamma)$, equipped with the induced norm. If Γ is non-amenable, then $H_{(p)}^1(\Gamma)$ is a Banach space.

Now let us turn again to non-elementary hyperbolic groups. Applying a beautiful result of Bourdon and Pajot [6, Thm.0.1, Thm.3.4] to our situation, we obtain the following fact:

Proposition 8.2. *Let Γ be a non-elementary hyperbolic group. Then the following hold:*

- i) *for each $\phi \in E_p(\Gamma)$, the boundary extension $\phi_\infty(\xi) = \lim_{x \rightarrow \xi} \phi(x)$ is a.e. defined on $\partial\Gamma$;*
- ii) *the boundary extension induces a Γ -equivariant Banach space isomorphism $H_{(p)}^1(\Gamma) \rightarrow B_p(\partial\Gamma)$, given by $[\phi] \mapsto [\phi_\infty]$.*

In light of the equivariant isomorphism between the first ℓ^p -cohomology group and the boundary Besov space, Proposition 8.1 can be stated as follows:

Theorem 8.3. *Let Γ be a non-elementary hyperbolic group, and let $p \geq 1$. Then the linear isometric action of Γ on $H_{(p)}^1(\Gamma)$ admits $g \mapsto [\langle g, \cdot \rangle]$ as a proper cocycle for all $p > \hat{e}(\Gamma)$.*

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